Two Problems in Mechanics (10 points)

Please read the general instructions in the separate envelope before you start this problem.

Part A. The Hidden Disk (3.5 points)

We consider a solid wooden cylinder of radius r_1 and thickness h_1. Somewhere inside the wooden cylinder, the wood has been replaced by a metal disk of radius r_2 and thickness h_2. The metal disk is placed in such a way that its symmetry axis B is parallel to the symmetry axis S of the wooden cylinder, and is placed at the same distance from the top and bottom face of the wooden cylinder, as shown in Fig. 1(b) below. We denote the distance between S and B by d. The density of wood is ρ_1, the density of the metal is $\rho_2 > \rho_1$. The total mass of the whole system is M.

In this task, we place the wooden cylinder on the ground so that it can freely roll to the left and right. See Fig. 1 for a side view and a view from the top of the setup.

The goal of this task is to determine the dimensions and the position of the metal disk.

In what follows, when asked to express the result in terms of known quantities, you may always assume that the following quantities are known, these are called quantities (1):

$$r_1,\ h_1,\ \rho_1,\ \rho_2, M\ . \tag{1}$$

The goal is to determine r_2, h_2 and d, through indirect measurements.

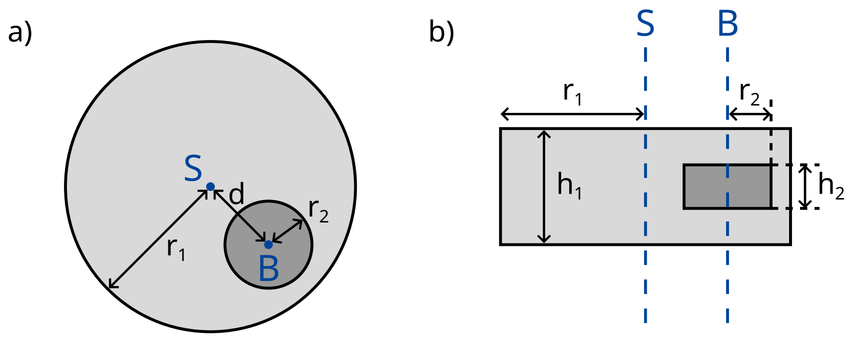

Figure 1: a) side view b) view from above

We denote b as the distance between the centre of mass C of the whole system and the symmetry axis S of the wooden cylinder. In order to determine this distance, we design the following experiment: We place the wooden cylinder on a horizontal base. Let us now slowly incline the base by an angle Θ (see Fig. 2). As a result of the static friction, the wooden cylinder is able to roll freely without sliding. It will roll down the incline a little bit, but then come to rest in a stable equilibrium after rotating by an angle ϕ which we measure.

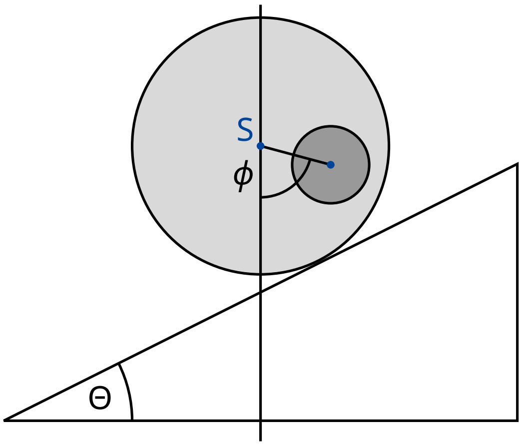

Figure 2: Cylinder on an inclined base.

A.1	Find an expression for b as a function of the quantities (1), the angle ϕ and the tilting angle Θ of the base.	0.8pt

From now on, we can assume that the value of b is known. Therefore, you may express your answers in terms of b.

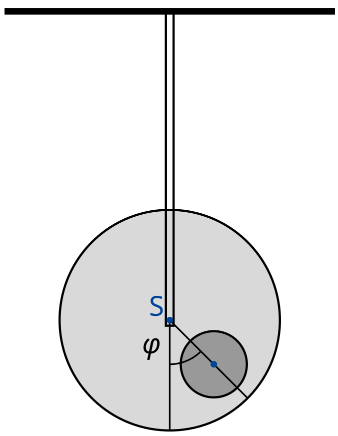

Figure 3: Suspended system.

Next we want to measure the moment of inertia I_S of the system with respect to the symmetry axis S. To do this, we suspend the wooden cylinder at its symmetry axis from a rigid rod that is immovable. We then displace the system away from its equilibrium position by a small angle φ, and release it. See figure 3 for the setup. We find that φ describes a periodic motion with period T i.e. this means that the system oscillates about the axis through S.

A.2	Write the equation of motion for φ. Express the moment of inertia I_S of the system around its symmetry axis S in terms of T, b and the known quantities (1). You may assume that φ is always very small.	0.5pt

From the measurements in questions **A.1** and **A.2**, we now want to determine the dimensions and its position of the metal disk inside the wooden cylinder.

A.3	Find an expression for the distance d in terms of b and the quantities (1). You may also include r_2 and h_2 as variables in your expression, as they will be calculated in subtask **A.5**.	0.4pt

A.4	Find an expression for the moment of inertia I_S in terms of b and the known quantities (1). You may also include r_2 and h_2 in your expression, as they will be calculated in subtask **A.5**.	0.7pt

A.5	Using all the above results, write down an expression for h_2 and r_2 in terms of b, T and the known quantities (1). You may express h_2 as a function of r_2.	1.1pt

Part B. Rotating Space Station (6.5 points)

Alice is an astronaut living on a space station. The space station is a gigantic wheel of radius R rotating around its axis, thereby providing artificial gravity for the astronauts. The astronauts live on the inner side of the rim of the wheel (see Fig. 4). The space station itself is so light that the gravitational attraction of the space station can be ignored.

B.1	At what angular frequency ω_{ss} does the space station have to rotate so that the astronauts experience the same gravity g_E as on the Earth's surface? (The radius R is very large compared to the size of the astronaut).	0.5pt

Alice and her astronaut friend Bob have an argument. Bob does not believe that they are in fact living in a space station and instead claims that they are on Earth. Alice wants to prove to Bob that they are living on a rotating space station by using physics. To do this, she attaches a mass m to a spring with spring constant k and lets it oscillate. The mass oscillates only in the vertical direction, and cannot move in the horizontal direction.

B.2	If Alice were to do perform this experiment on Earth, with gravitational acceleration g_E, what is the angular oscillation frequency ω_E?	0.2pt

B.3	Alice repeats the experiment on the space station. What is the angular oscillation frequency ω now?	0.6pt

Alice is convinced that her experiment proves that they are on a rotating space station. However Bob remains unconvinced. He claims that a similar result will be obtained if the experiment is carried out above the surface of the Earth by taking into account the change in gravity above the Earth's surface. In the following tasks we investigate whether Bob is right.

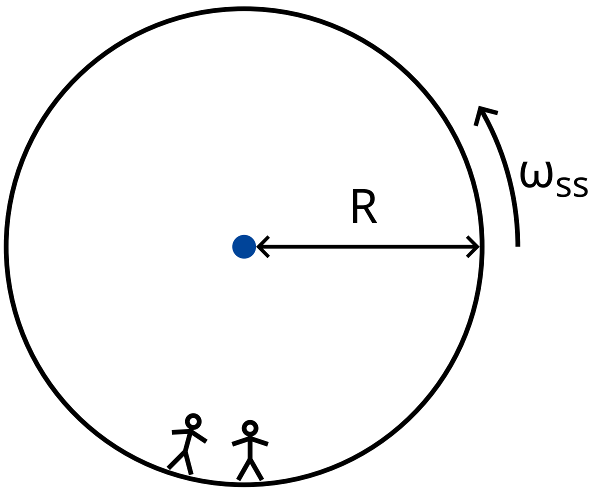

Figure 4: Space station

B.4	Derive an expression of the gravitational acceleration $g_E(h)$ for small heights h above the surface of the Earth and compute the oscillation frequency $\tilde{\omega}_E$ of the oscillating mass (linear approximation is enough). Denote the radius of the Earth by R_E. Neglect the rotation of Earth.	0.8pt

Indeed, Alice does find that the spring oscillator oscillates with the frequency that Bob predicted on this space station.

B.5	For what radius R of the space station will the oscillation frequency ω matches with the oscillation frequency $\tilde{\omega}_E$ on the Earth? Express your answer in terms of R_E.	0.3pt

Frustrated with Bob's stubbornness, Alice comes up with another experiment to prove her point. To do this, she climbs on a tower of height H over the floor of the space station and drops a mass. This experiment can be understood in the rotating reference frame as well as in an inertial reference frame.

In a uniformly rotating reference frame, the astronauts perceive a fictitious force $\vec{F}_C$ called the Coriolis force. The Coriolis force $\vec{F}_C$ acting on an object of mass m moving at velocity $\vec{v}$ in a rotating frame with constant angular frequency $\vec{\omega}_{ss}$ is given by

$$\vec{F}_C = 2m\vec{v} \times \vec{\omega}_{ss} . \tag{2}$$

In terms of the scalar quantities you may use

$$F_C = 2mv\omega_{ss} \sin\phi , \tag{3}$$

where ϕ is the angle between the velocity and the axis of rotation. The force is perpendicular to both the velocity v and the axis of rotation. The direction of the force can be determined from the right-hand rule, but in what follows you may choose it freely.

B.6	Calculate the horizontal velocity v_x and the horizontal displacement d_x (relative to the base of the tower, in the direction perpendicular to the tower) of the mass at the moment it hits the floor. You may assume that the height H of the tower is small, so that the acceleration as measured by the astronauts is constant during the fall. Also, you may assume that $d_x \ll H$.	1.1pt

To get a good result, Alice decides to conduct this experiment from a much taller tower than before. To her surprise, the mass hits the floor at the base of the tower i.e. $d_x = 0$.

B.7	Find a lower bound for the height of the tower for which it can happen such that $d_x = 0$.	1.3pt

Alice is willing to make one last attempt at convincing Bob. She wants to use her spring oscillator to show the effect of the Coriolis force. To do this, she changes the original setup: She attaches her spring to a ring which can slide freely on a horizontal rod in the x direction without any friction. The spring itself oscillates in the y direction. The rod is parallel to the floor and perpendicular to the axis of rotation of the space station. The xy plane is thus perpendicular to the axis of rotation, with the y direction pointing straight towards the center of rotation of the station.

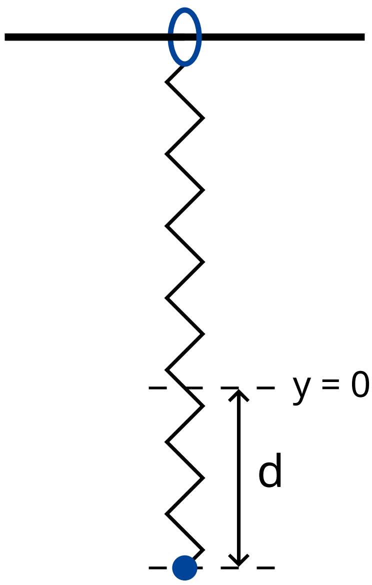

Figure 5: Setup.

B.8	Alice pulls the mass a distance d downwards from the equilibrium point $x = 0$, $y = 0$, and then lets it go (see figure 5). • Give an algebraic expression of $x(t)$ and $y(t)$. You may assume that $\omega_{ss} d$ is small, and neglect the Coriolis force for motion along the y-axis. • Sketch the trajectory $(x(t), y(t))$ of the mass, marking all important features such as amplitude.	1.7pt

Alice and Bob continue to argue.