Two Problems in Mechanics. Ду масъала аз Механика (10 points/холл)

Please read the general instructions in the separate envelope before you start this problem.

Пеш аз ин масъаларо сар кардан қоидаҳои умумие, ки дар конверти алоҳида дода мешаванд бодиққат хонед.

Part A. The Hidden Disk (3.5 points)

We consider a solid wooden cylinder of radius r_1 and thickness h_1. Somewhere inside the wooden cylinder, the wood has been replaced by a metal disk of radius r_2 and thickness h_2. The metal disk is placed in such a way that its symmetry axis B is parallel to the symmetry axis S of the wooden cylinder, and is placed at the same distance from the top and bottom face of the wooden cylinder. We denote the distance between S and B by d. The density of wood is ρ_1, the density of the metal is $\rho_2 > \rho_1$. The total mass of the wooden cylinder and the metal disk inside is M.

Мо як силидри чӯбини радиусаш r_1 ва бараш h_1 дорем. Куҷое дар даруни ин силиндр диски металии радиусаш r_2 ва бараш h_2 гузошта шудааст. Диски металӣ ба тавре гузошта шудааст, ки тири (меҳвари) симметриаш B бо тири (меҳвари) симметрии силиндр S параллел мебошад. Ва ин диск аз ду рӯйи (тагу рӯйи) силиндр дар масофаи якхел дур гузошта шудааст. Мо масофаи байни S ва B -ро бо d ишорат мекунем. Зичии чӯб (тахта) ρ_1 ва зичии металро $\rho_2 > \rho_1$ қабул кунед. Массаи умумии силиндри чӯбӣ ва диски металии дар дарунбуда M қабул кунед.

In this task, we place the cylinder on the ground so that it can freely roll to the left and right. See Fig. 1 for a side view and a view from the top of the setup.

Дар ин қисм, мо силиндрро ба рӯйи замин мегузорем, ки дар он ба чапу рост ғелида (тобхӯрда) метавонад. Барои тасвири паҳлӯӣ ва болоии система нигаред ба расми 1.

The goal of this task is to determine the size and the position of the metal disk.

Мақсади ин қисмат муайян кардани андоза ва мавқеъи диски металӣ мебошад.

When asked to express the result in terms of known quantities, you may always assume that the followings are known:

Дар давоми имтиҳон ҳангоми натиҷаҳоро аз тариқи дигар бузургиҳои (қимматҳои) физикӣ навистан метавонед қабул кард, ки бузургиҳои зерин дониста меашаванд:

$$r_1,\ w_1,\ \rho_1,\ \rho_2, M\ . \qquad (1)$$

The goal is to determine r_2, h_2 and d, through indirect measurements.

Мақсад муайян кардани r_2, w_2 ва d тавассути бавосита ченкунӣ (андозагирӣ) мебошад.

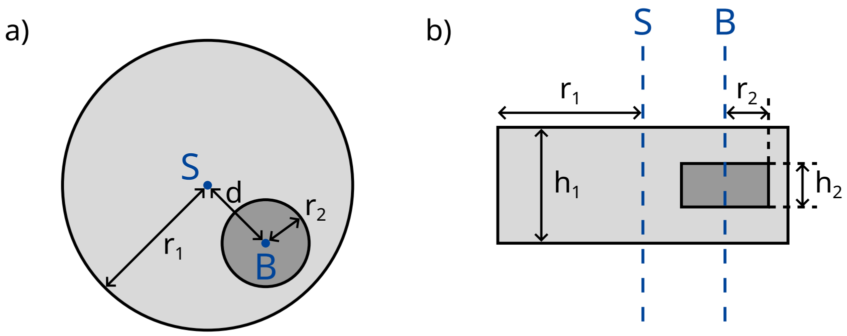

Figure 1: a) side view *(тасвири паҳлўӣ)* b) view from top *(тасвири болоӣ)*

We denote b as the distance between the centre of mass C of the whole system and the symmetry axis S of the wooden cylinder. In order to determine this distance, we design the following experiment: We place the wooden cylinder on a horizontal base in such a way that it is in a stable equilibrium. Let us now slowly incline the base by an angle Θ (see Fig. 2). As a result of the static friction, the wooden cylinder can roll freely without sliding. It will roll down the incline a little bit, but then come to rest in a stable equilibrium after rotating by an angle ϕ which we measure.

b масофаи байни маркази массаи тамоми система C ва тири симметрияи силиндр S мебошад. Барои ин масофаро муайян кардан мо эксперименти мазкурро месозем: Силиндрро дар болои ҳамвории амудӣ (горизонталӣ) бо мувозинати устувор мегузорем. Баъдан аз як тарафи ҳамворӣ бо миқдори кунҷи Θ майл медиҳем (мебардорем). Нигаред ба расми 2. Аз сабаби будани соиши статикӣ силиндр налағжида ғелида (ҷар шуда) метавонад. Он дар болои нишебии моил каме меғелад, лекин баъд боз пас аз ба миқдори кунҷи ϕ давр задан ба мувозинати устувор меояд, ки мо чен карданием.

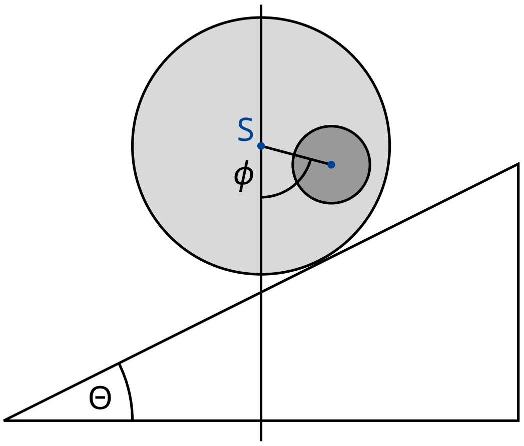

Figure 2: Cylinder on an inclined base. (*Силиндр дар болои нишебии моил)*

A.1 **Find an expression for b as a function of the quantities (1), the angle ϕ and the tilting angle Θ of the base.** 0.8pt
Барои муайян намудани b ифодаеро аз тариқи бузургиҳои (1), кунҷи ϕ ва кунҷи тобхурии Θ-и нишебӣ ёбед.

From now on, we can assume that the value of b is known.

Аз ин пас мо метованем қабул кунем, ки қиммати b дониста мешавад.

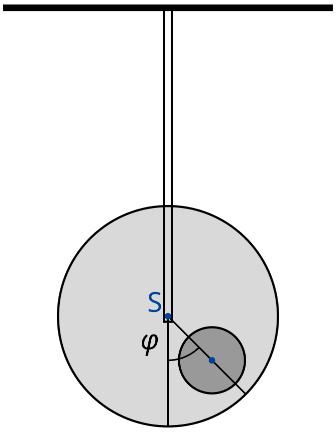

Figure 3: Suspended cylinder. *(силиндри овезон)*

Next we want to measure the moment of inertia I_S of the system with respect to the symmetry axis S. To this end, we suspend the wooden cylinder at its symmetry axis from a rigid rod. We then turn it away from its equilibrium position by a small angle φ, and let it go. See figure 3 for the setup. We find that φ describes a periodic motion with period T.

Дар давом мо мехоҳем моменти инертсиалии силиндр I_S-ро назар ба тири симетрии S чен кунем. То ин ҷо мо силиндрро аз тири симметриаш овезон карда будем. Баъдан мо онро аз ҳолати мувозинат ба миқдори кунҷи хурди φ меҷунбонем (мебардорем) ва сар медиҳем. Барои маълумоти бештар нигаред ба расми 3. Ба мо маълум мегардад, ки φ ҳаракати davриро (такрорӣ) нишон медиҳад, ки периодаш T мебошад.

A.2 **Find the equation of motion for φ. Express the moment of inertia I_S of the system around its symmetry axis S in terms of T, b and the known quantities (1). You may assume that we are only disturbing the equilibrium position by a small amount so that φ is always very small.** 0.5pt
Ифодаи ҳаркати φ-ро ёбед. Моменти инетсиалӣ I_S-и системаро дар гирди тири симметрияш S аз тариқи T, b ва бузургиҳои маълуми (1) ёбед. Дар ёд дошта бошед, ки мо ҳолати мувозинатро фақат бо миқдори хурди φ меҷунбонем ва он ҳамеша хело хурд мебошад.

From the measurements in questions **A.1** and **A.2**, we now want to determine the geometry and the position of the metal disk inside the wooden cylinder.

A.3	Find an expression for the distance d as a function of b and the quantities (1). You may also include r_2 and h_2 as variables in your expression, as they will be calculated in subtask **A.5**.	0.4pt

A.4	Find an expression for the moment of inertia I_S in terms of b and the known quantities (1). You may also include r_2 and h_2 as variables in your expression, as they will be calculated in subtask **A.5**.	0.7pt

A.5	Using all the above results, write down an expression for h_2 and r_2 in terms of b, T and the known quantities (1). You may express h_2 as a function of r_2.	1.1pt

Part B. Rotating Space Station (6.5 points)

Alice is an astronaut living on a space station. The space station is a gigantic wheel of radius R rotating around its axis, thereby providing artificial gravity for the astronauts. The astronauts live on the inner side of the rim of the wheel. The gravitational attraction of the space station and the curvature of the floor can be ignored.

B.1	At what angular frequency ω_{ss} does the space station rotate so that the astronauts experience the same gravity g_E as on the Earth's surface?	0.5pt

Alice and her astronaut friend Bob have an argument. Bob does not believe that they are in fact living in a space station and claims that they are on Earth. Alice wants to prove to Bob that they are living on a rotating space station by using physics. To this end, she attaches a mass m to a spring with spring constant k and lets it oscillate. The mass oscillates only in the vertical direction, and cannot move in the horizontal direction.

B.2	Assuming that on Earth gravity is constant with acceleration g_E, what would be the angular oscillation frequency ω_E that a person on Earth would measure?	0.2pt

B.3	What angular oscillation frequency ω does Alice measure on the space station?	0.6pt

Alice is convinced that her experiment proves that they are on a rotating space station. Bob remains sceptical. He claims that when taking into account the change in gravity above the surface of the Earth, one finds a similar effect. In the following tasks we investigate whether Bob is right.

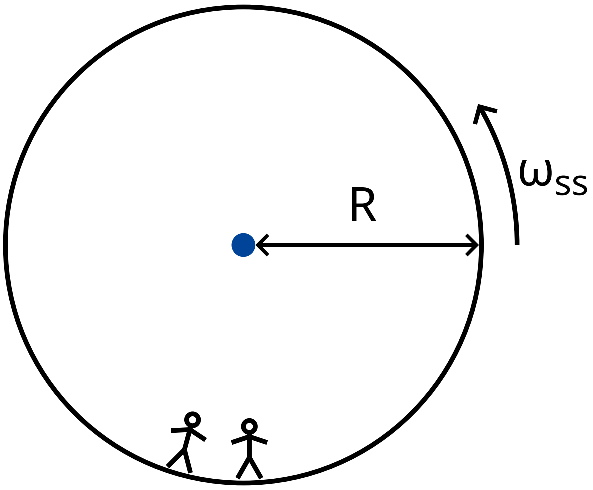

Figure 4: Space station

B.4	Derive an expression of the gravity $g_E(h)$ for small heights h above the surface of the Earth and compute the oscillation frequency $\tilde{\omega}_E$ of the oscillating mass (linear approximation is enough). Denote the radius of the Earth by R_E. Neglect the rotation of Earth.	0.8pt

Indeed, for this space station, Alice does find that the spring pendulum oscillates with the frequency that Bob predicted.

B.5	For what radius R of the space station does the oscillation frequency ω match the oscillation frequency $\tilde{\omega}_E$ on the Earth? Express your answer in terms of R_E.	0.3pt

Exasperated with Bob's stubbornness, Alice comes up with an experiment to prove her point. To this end she climbs on a tower of height H over the floor of the space station and drops a mass. This experiment can be understood in the rotating reference frame as well as in an inertial reference frame.

In a uniformly rotating reference frame, the astronauts perceive a fictitious force $\vec{F}_C$ called the Coriolis force. The force $\vec{F}_C$ acting on an object of mass m moving at velocity $\vec{v}$ in a rotating frame with constant angular frequency $\vec{\omega}_{ss}$ is given by

$$\vec{F}_C = 2m\vec{v} \times \vec{\omega}_{ss} \,. \tag{2}$$

In terms of the scalar quantities you may use

$$F_C = 2mv\omega_{ss} \sin\phi \,, \tag{3}$$

where ϕ is the angle between the velocity and the axis of rotation. The force is perpendicular to both the velocity v and the axis of rotation. The sign of the force can be determined from the right-hand rule, but in what follows you may choose it freely.

B.6	Calculate the horizontal velocity v_x and the horizontal displacement d_x (relative to the base of the tower, in the direction perpendicular to the tower) of the mass at the moment it hits the floor. You may assume that the height H of the tower is small, so that the acceleration as measured by the astronauts is constant during the fall. Also, you may assume that $d_x \ll H$.	1.1pt

To get a good result, Alice decides to conduct this experiment from a much taller tower than before. To her surprise, the mass hits the floor at the base of the tower, so that $d_x = 0$.

B.7	Find a lower bound for the height of the tower for which it can happen that $d_x = 0$.	1.3pt

Alice is willing to make one last attempt at convincing Bob. She wants to use her spring oscillator to show the effect of the Coriolis force. To this end she changes the original setup: She attaches her spring to a ring which can slide freely on a horizontal rod in the x direction without any friction. The spring itself oscillates in the y direction. The rod is parallel to the floor and perpendicular to the axis of rotation of the space station. The xy plane is thus perpendicular to the axis of rotation, with the y direction pointing straight towards the center of rotation of the station.

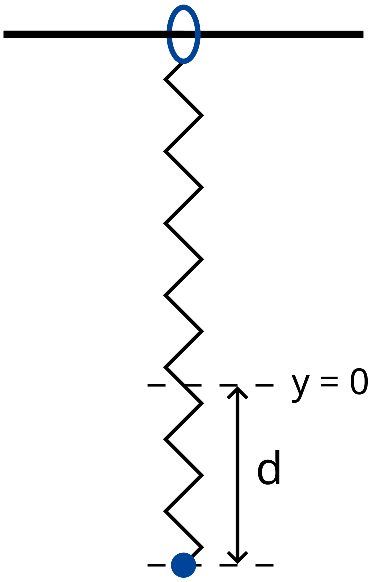

Figure 5: Setup.

B.8	Alice pulls the mass a distance d downwards from the equilibrium point $x = 0$, $y = 0$, and then lets it go (see figure 5). • Give an algebraic expression of $x(t)$ and $y(t)$. You may assume that $\omega_{ss} d$ is small, and neglect the Coriolis force for motion along the y-axis. • Sketch the trajectory $(x(t), y(t))$, marking all important features such as amplitude.	1.7pt

Alice and Bob continue to argue.