Nonlinear Dynamics in Electric Circuits (10 points)

Please read the general instructions in the separate envelope before you start this problem.

Introduction

Bistable non-linear semiconducting elements (e.g. thyristors 晶閘體) are widely used in electronics as switches and generators of electromagnetic oscillations. The primary field of applications of thyristors is controlling alternating currents in power electronics, for instance rectification of AC current to DC at the megawatt scale. Bistable elements may also serve as model systems for self-organization phenomena in physics (this topic is covered in part B of the problem), biology (see part C) and other fields of modern nonlinear science.

Goals

To study instabilities and nontrivial dynamics of circuits including elements with non-linear $I-V$ characteristics. To discover possible applications of such circuits in engineering and in modeling of biological systems.

Part A. Stationary states and instabilities (3 points) 平穩狀態及其不穩定性

Fig. 1 shows the so-called **S-shaped** $I-V$ characteristics of a non-linear element X. In the voltage range between $U_\mathrm{h} = 4.00\ \mathrm{V}$ (the holding voltage) and $U_\mathrm{th} = 10.0\ \mathrm{V}$ (the threshold voltage) this $I-V$ characteristics is multivalued. For simplicity, the graph on Fig. 1 is chosen to be piece-wise linear (each branch is a segment of a straight line). In particular, the line in the upper branch touches the origin if it is extended. This approximation gives a good description of real thyristors.

圖 1 為非線性部件 X 之 S 型 $I-V$ 圖。當電壓 V 介乎 $U_\mathrm{h} = 4.00\ \mathrm{V}$ 至 $U_\mathrm{th} = 10.0\ \mathrm{V}$ 時，一個電壓值會對應多於一個電流值。為簡化問題，假設每一截關係都是線性的。最上一截之直線圖如果向下伸延，會通過原點。以上之假設對現實晶閘體是足夠好的。

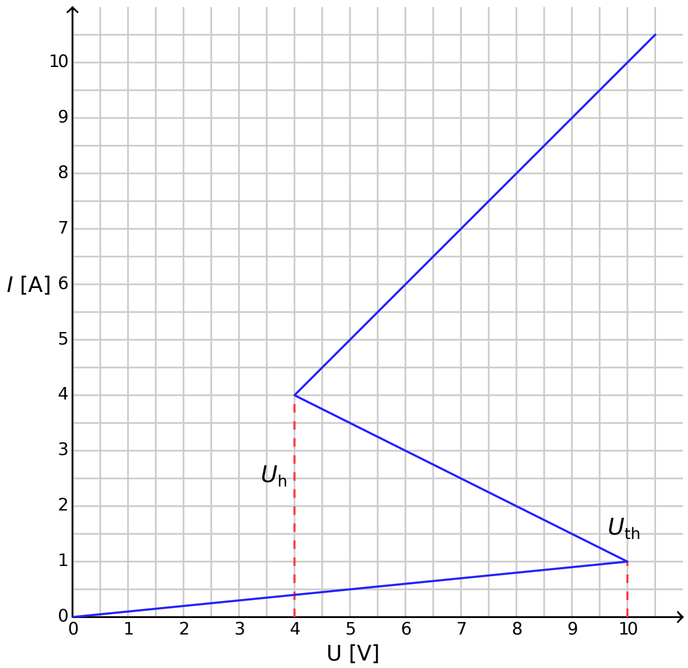

Figure 1（圖 1）：$I-V$ characteristics of the non-linear element X.

A.1	Using the graph, determine the resistance R_{on} of the element X on the upper branch of the $I-V$ characteristics, and R_{off} on the lower branch, respectively. The middle branch is described by the equation 根據所供給的圖，找出最上一截 $I-V$ 關係中的電阻 R_{on}，及最下一截 $I-V$ 關係中的電阻 R_{off}。中間一截 $I-V$ 關係可用以下方程描述： $$I = I_0 - \frac{U}{R_{\text{int}}}. \quad (1)$$ Find the values of the parameters I_0 and R_{int}. 找出方程中之參數 I_0 及 R_{int}。	0.4pt

The element X is connected in series (see Fig.2) with a resistor R, an inductor L and an ideal voltage source $\mathcal{E}$. One says that the circuit is in a stationary state if the current is constant in time, $I(t) = \text{const}$.

將部件 X、電阻 R、電感 L 及理想電壓源 $\mathcal{E}$ 串聯接駁。如果最終電流隨時間不變，$I(t) = \text{const}$，我們稱之為平穩狀態。

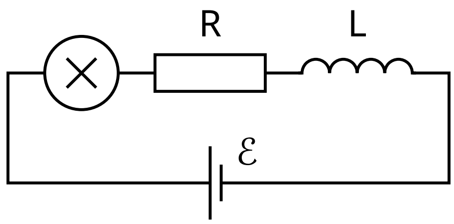

Figure 2（圖 2）：Circuit with element X, resistor R, inductor L and voltage source $\mathcal{E}$.

A.2	What are the possible numbers of stationary states that the circuit of Fig. 2 may have for a fixed value of $\mathcal{E}$ and for $R = 3.00\ \Omega$? How does the answer change for $R = 1.00\ \Omega$? 對於 $R = 3.00\ \Omega$，平穩狀態的總數可以是甚麼值？對於 $R = 1.00\ \Omega$ ，平穩狀態的總數可以是甚麼值？	1pt

A.3	Let $R = 3.00\ \Omega, L = 1.00\ \mu\text{H}$ and $\mathcal{E} = 15.0\ \text{V}$ in the circuit shown in Fig. 2. Determine the values of the current $I_{\text{stationary}}$ and the voltage $V_{\text{stationary}}$ on the non-linear element X in the stationary state. 設圖 2 中 $R = 3.00\ \Omega, L = 1.00\ \mu\text{H}$ 及 $\mathcal{E} = 15.0\ \text{V}$。找出平穩狀態下流過 X 之電流 $I_{\text{stationary}}$ 及跨過 X 之電壓 $V_{\text{stationary}}$。	0.6pt

The circuit in Fig. 2 is in the stationary state with $I(t) = I_{\text{stationary}}$. This stationary state is said to be stable if after a small displacement (increase or decrease in the current), the current returns towards the stationary state. And if the system keeps moving away from the stationary state, it is said to be unstable.

現圖 2 處於平穩狀態，$I(t) = I_{\text{stationary}}$。如果狀態經歷微擾後（電流上升或下降），電流會回復原先狀態，這個平穩狀態就稱為　穩定；但如困經微擾後，系統會遠離原先狀態，此平穩狀態就稱為　不穩定。

A.4	Use numerical values of the question **A.3** and study the stability of the stationary state with $I(t) = I_{\text{stationary}}$. Is it stable or unstable? 用 A.3 之數值去研究該平穩狀態 $I(t) = I_{\text{stationary}}$ 是否穩定。	1pt

Part B. Bistable non-linear elements in physics: radio transmitter (5 points) 物理上之應用：無線電波發射器

We now investigate a new circuit configuration (see Fig. 3). This time, the non-linear element X is connected in parallel to a capacitor of capacitance $C = 1.00\ \mu\text{F}$. This block is then connected in series to a resistor of resistance $R = 3.00\ \Omega$ and an ideal constant voltage source of voltage $\mathcal{E} = 15.0\ \text{V}$. It turns out that this circuit undergoes oscillations with the non-linear element X jumping from one branch of the $I - V$ characteristics to another over the course of one cycle.

現研究一新情況(圖 3)。這次，X 與電容 $C = 1.00\ \mu\text{F}$ 並聯。這組合與電阻 $R = 3.00\ \Omega$ 及理想電壓源 $\mathcal{E} = 15.0\ \text{V}$ 串聯。在這特定情況下，電路出現周期性振動循環，當中 X 在一循環內，出現由其中一截 $I - V$ 關係跳到另一截 $I - V$ 關係之變化。

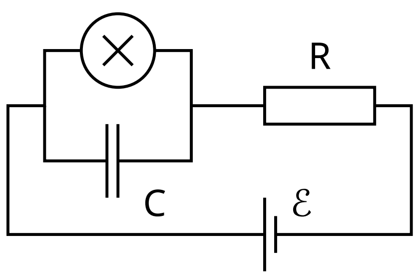

Figure 3（圖 3): Circuit with element X, capacitor C, resistor R and voltage source $\mathcal{E}$.

B.1	Draw the oscillation cycle on the $I-V$ graph, including its direction (clockwise or anticlockwise). Justify your answer with equations and sketches. 在 $I-V$ 圖中畫出此振動循環。標出它的方向（順時針或逆時針）。用數式及繪圖去解釋你的答案。	1.8pt

B.2	Find expressions for the times t_1 and t_2 that the system spends on each branch of the $I-V$ graph during the oscillation cycle. Determine their numerical values. Find the numerical value of the oscillation period T assuming that the time needed for jumps between the branches of the $I-V$ graph is negligible. 在此循環周期內，電路花了時間 t_1 在其中一截 $I-V$ 關係，並花了時間 t_2 在另一截 $I-V$ 關係。找出 t_1 及 t_2 之表達式。找出它們之數值。也找出循環周期之時間值 T。可忽略電路在每截 $I-V$ 關係之間之跳躍時間。	1.9pt

B.3	Estimate the average power P dissipated by the non-linear element over the course of one oscillation. An order of magnitude is sufficient. 估計此非線性部份在一循環周期內所消耗之平均功率 P。給出數量級。	0.7pt

The circuit in Fig. 3 is used to build a radio transmitter. For this purpose, the element X is attached to one end of a linear antenna (a long straight wire) of length s. The other end of the wire is free. In the antenna, an electromagnetic standing wave is formed. The speed of electromagnetic waves along the antenna is the same as in vacuum. The transmitter is using the main harmonic of the system, which has period T of question **B.2**.

Fig.3 之電路現用於一無線電發射器。把 X 連接在一長度為 s 之直線天線之一端。另一端是一個自由端。沿著天線，電磁波會以駐波形式出現。電磁波在天線之傳送速度與真空一樣。發射器是以天線系統之基本和諧頻率發射無線電波，其無線電波之周期為 T (公式 B.2)。

B.4	What is the optimal value of s assuming that it cannot exceed $1\ \mathrm{km}$? 假如 s 不超過 $1\ \mathrm{km}$，找出 s 最理想的值。	0.6pt

Part C. Bistable non-linear elementss in biology: neuristor (2 points) 生物學之應用：模擬神經元

In this part of the problem, we consider an application of bistable non-linear elements to modeling of biological processes. A neuron in a human brain has the following property: when excited by an external signal, it makes one single oscillation and then returns to its initial state. This feature is called excitability. Due to this property, pulses can propagate in the network of coupled neurons constituting the nerve systems. A semiconductor chip designed to mimic excitability and pulse propagation is called a neuristor (from neuron and transistor).

人體的神經元當受外界刺激，會振動一次，並把脈衝傳到神經網絡中附近之神經元。以半導體去模擬這性質，就稱為模擬神經元。

We attempt to model a simple neuristor using a circuit that includes the non-linear element X that we investigated previously. To this end, the voltage $\mathcal{E}$ in the circuit of Fig. 3 is decreased to the value $\mathcal{E}' = 12.0\ \mathrm{V}$. The oscillations stop, and the system reaches its stationary state. Then, the voltage is rapidly increased back to the value $\mathcal{E} = 15.0\ \mathrm{V}$, and after a period of time τ (with $\tau < T$) is set again to the value $\mathcal{E}'$ (see Fig. 4). It turns out that there is a certain critical value $\tau_{\mathrm{crit.}}$, and the system shows qualitatively different behavior for $\tau < \tau_{\mathrm{crit}}$ and for $\tau > \tau_{\mathrm{crit}}$.

為達以上目的：Fig.3 的電壓 $\mathcal{E}$ 降為 $\mathcal{E}' = 12.0\ \mathrm{V}$。在此情況下，電路系統之振動循環不會發生，並達至一平穩

狀態。跟著，電壓迅速回升至 $\mathcal{E} = 15.0\ \mathrm{V}$，並在時間 $\tau(\tau < T)$ 後跳回 $\mathcal{E}'$(Fig. 4)。我們發現一臨界時間值 $\tau_{\mathrm{crit.}}$，$\tau < \tau_{\mathrm{crit}}$ 或 $\tau > \tau_{\mathrm{crit}}$，電路系統的表現有差異。

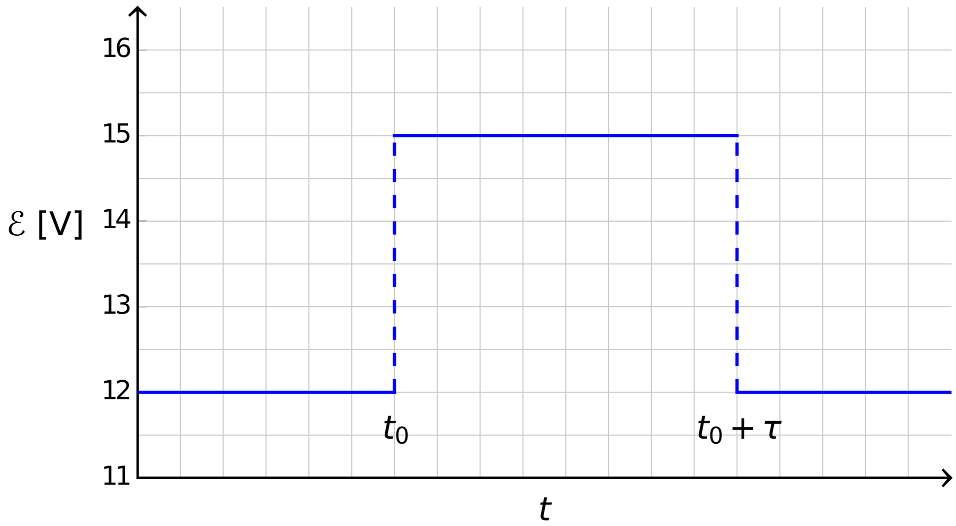

Figure 4: Voltage of the voltage source as a function of time.

C.1	Sketch the graphs of the time dependence of the current $I_X(t)$ on the non-linear element X for $\tau < \tau_{\mathrm{crit}}$ and for $\tau > \tau_{\mathrm{crit}}$. 對於 $\tau < \tau_{\mathrm{crit}}$ 及 $\tau > \tau_{\mathrm{crit}}$，分別畫出流過部件 X 之電流 $I_X(t)$ 隨時間之變化圖。	1.2pt

C.2	Find the expression and the numerical value of the critical time τ_{crit} for which the scenario switches. 找出臨界時間值 τ_{crit} 之表達式及其數值。	0.6pt

C.3	Is the circuit with $\tau = 1.00 \times 10^{-6}\ \mathrm{s}$ a neuristor? 如果 $\tau = 1.00 \times 10^{-6}\ \mathrm{s}$，這電路可用作模擬神經元嗎？	0.2pt